Uniqueness theorem for an inverse scattering problem with non-overdetermined data

2010 J. Phys. A: Math. Theor. 43 112001

(http://iopscience.iop.org/1751-8121/43/11/112001)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 112001 (11pp) doi:10.1088/1751-8113/43/11/112001

FAST TRACK COMMUNICATION

Uniqueness theorem for an inverse scattering problem with non-overdetermined data

A G Ramm

Department of Mathematics, Kansas State University, Manhattan, KS 66506-2602, USA

E-mail: ramm@math.ksu.edu

Received 30 October 2009, in final form 12 January 2010
Published 26 February 2010
Online at stacks.iop.org/JPhysA/43/112001

Abstract
Let $q(x)$ be a real-valued compactly supported sufficiently smooth function. It is proved that the scattering data $A(-\beta, \beta, k)$ $\forall\beta \in S^2$, $\forall k > 0$ determine q uniquely.

PACS number: 03.65.Nk
Mathematics Subject Classification: 35P25, 35R30, 81Q05

1. Introduction

Since the forties of the last century, the physicists were interested in the uniqueness of the determination of a physical system by its S-matrix. If the physical system is described by a Hamiltonian of the type $H = -\nabla^2 + q(x)$, then the S-matrix is in one-to-one correspondence with the scattering amplitude A, $S = I + \frac{ik}{2\pi}A$, where I is the identity operator and A is an operator in $L^2(S^2)$ with the kernel $A(\beta, \alpha, k)$, S^2 is the unit sphere in $\mathbb{R}^3$ and k^2 is the energy, $k > 0$. The scattering amplitude is defined as follows. If the incident plane wave $u_0 = e^{ik\alpha\cdot x}$, $\alpha \in S^2$, is scattered by the potential q, then the scattering solution $u(x, \alpha, k)$ solves the scattering problem:

$$[\nabla^2 + k^2 - q(x)]u = 0 \quad \text{in} \quad \mathbb{R}^3 \tag{1}$$

$$u = e^{ik\alpha\cdot x} + A(\beta, \alpha, k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \qquad r := |x| \to \infty, \quad \beta := \frac{x}{r}. \tag{2}$$

The coefficient $A(\beta, \alpha, k)$ is called the scattering amplitude. The problem of interest is to determine $q(x)$ given the scattering data. This problem is called the inverse scattering problem. The function $A(\beta, \alpha, k)$ depends on two unit vectors β, α, and on the scalar k, i.e. on five variables.

Assumption (A). *We assume that q is compactly supported, i.e. $q(x) = 0$ for $|x| > a$, where $a > 0$ is an arbitrary large fixed number, $q(x)$ is real valued, i.e. $q = \overline{q}$, and $q(x) \in H_0^\ell(B_a)$, $\ell > 2$.*

Here B_a is the ball centered at the origin and of radius a, and $H_0^\ell(B_a)$ is the closure of $C_0^\infty(B_a)$ in the norm of the Sobolev space $H^\ell(B_a)$ of functions whose derivatives up to the order ℓ belong to $L^2(B_a)$. It was proved in [4] (see also [5] and [6], chapter 6) that if $q = \overline{q}$ and $q \in L^2(B_a)$ is compactly supported, then the resolvent kernel $G(x, y, k)$ of the Schrödinger operator $-\nabla^2 + q(x) - k^2$ is a meromorphic function of k on the whole complex plane k, analytic in $\mathrm{Im}k \geqslant 0$ except, possibly, of a finitely many simple poles at the points $\mathrm{i}k_j$, $k_j > 0$, $1 \leqslant j \leqslant n$, where $-k_j^2$ are negative eigenvalues of the self-adjoint operator $-\nabla^2 + q(x)$ in $L^2(\mathbb{R}^3)$. Consequently, the scattering amplitude $A(\beta, \alpha, k)$, corresponding to the above q, is a restriction to the positive semi-axis $k \in [0, \infty)$ of a meromorphic on the whole complex k-plane function.

It was known long ago that the scattering data $A(\beta, \alpha, k)$ $\forall \alpha, \beta \in S^2$, $\forall k > 0$, determine $q(x)$ uniquely. For even larger class of potentials, this result was proved in [12].

The above scattering data depend on five variables (two unit vectors α and β, and one scalar k). The potential $q(x)$ depends on three variables (x_1, x_2, x_3). Therefore, the inverse scattering problem, which consists of finding q from the above scattering data, is overdetermined.

It was proved by the author [7] that the *fixed-energy scattering data* $A(\beta, \alpha) := A(\beta, \alpha, k_0)$, $k_0 = \text{const} > 0$, $\forall \beta \in S_1^2$, $\forall \alpha \in S_2^2$, determine real-valued compactly supported $q \in L^2(B_a)$ uniquely. Here S_j^2, $j = 1, 2$, are arbitrary small open subsets of S^2 (solid angles). In [8] (and in [9], chapter 5) an analytical formula is derived for the reconstruction of q from the exact fixed-energy scattering data, and from noisy fixed-energy scattering data, and stability estimates for the reconstruction method are obtained.

The scattering data $A(\beta, \alpha)$ depend on four variables (two unit vectors), while the unknown $q(x)$ depends on three variables. In this sense the inverse scattering problem, which consists of finding q from the fixed-energy scattering data $A(\beta, \alpha)$, is still overdetermined.

For many decades there were no uniqueness theorems for 3D inverse scattering problems with non-overdetermined data. The goal of this communication is to prove such a theorem.

Theorem 1.1. *If $\overline{q} = q \in H_0^\ell(B_a)$, $\ell > 2$, then the data $A(-\beta, \beta, k)$ $\forall \beta \in S^2$, $\forall k > 0$, determine q uniquely.*

Remark 1. The conclusion of theorem 1.1 remains valid if the data $A(-\beta, \beta, k)$ are known $\forall \beta \in S_1^2$ and $k \in (k_0, k_1)$ where $(k_0, k_1) \subset [0, \infty)$ is an arbitrary small interval, $k_1 > k_0$, and S_1^2 is an arbitrary small open subset of S^2.

In some physical problems, the potential may depend on k, $q = q(x, k)$, $x \in R^3$, $k \in [0, \infty)$. In this case the inverse scattering problem with the back-scattering data $A(-\beta, \beta, k)$ is underdetermined: the data is a function of three variables while $q(x, k)$ depends on four variables. In general, one cannot expect that this inverse scattering problem has a unique solution.

In section 2 we formulate some known auxiliary results. In section 3 proof of theorem 1.1 is given. In the appendix a technical estimate is proved. A brief announcement of the result, stated in theorem 1.1, is given in [3].

2. Auxiliary results

Let

$$F(g) := \tilde{g}(\xi) = \int_{\mathbb{R}^3} g(x)\, \mathrm{e}^{\mathrm{i}\xi\cdot x}\, \mathrm{d}x, \quad g(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}\xi\cdot x}\, \tilde{g}(\xi)\, \mathrm{d}\xi. \tag{3}$$

If $f * g := \int_{\mathbb{R}^3} f(x-y)g(y)\,\mathrm{d}y$, then

$$F(f*g) = \tilde{f}(\xi)\tilde{g}(\xi), \qquad F(f(x)g(x)) = \frac{1}{(2\pi)^3}\tilde{f} * \tilde{g}. \tag{4}$$

If

$$G(x-y,k) := \frac{\mathrm{e}^{\mathrm{i}k[|x-y|-\beta\cdot(x-y)]}}{4\pi|x-y|}, \tag{5}$$

then

$$F(G(x,k)) = \frac{1}{\xi^2 - 2k\beta\cdot\xi}, \qquad \xi^2 := \xi\cdot\xi. \tag{6}$$

The scattering solution $u = u(x,\alpha,k)$ solves (uniquely) the integral equation

$$u(x,\alpha,k) = \mathrm{e}^{\mathrm{i}k\alpha\cdot x} - \int_{B_a} g(x,y,k)q(y)u(y,\alpha,k)\,\mathrm{d}y, \tag{7}$$

where

$$g(x,y,k) := \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|}. \tag{8}$$

If

$$v = \mathrm{e}^{-\mathrm{i}k\alpha\cdot x}u(x,\alpha,k), \tag{9}$$

then

$$v = 1 - \int_{B_a} G(x-y,k)q(y)v(y,\alpha,k)\,\mathrm{d}y, \tag{10}$$

where G is defined in (5).

Define ϵ by the formula

$$v = 1 + \epsilon. \tag{11}$$

Then (10) can be rewritten as

$$\epsilon(x,\alpha,k) = -\int_{\mathbb{R}^3} G(x-y,k)q(y)\,\mathrm{d}y - T\epsilon, \tag{12}$$

where

$$T\epsilon := \int_{B_a} G(x-y,k)q(y)\epsilon(y,\alpha,k)\,\mathrm{d}y.$$

The Fourier transform of (12) yields (see (4), (6))

$$\tilde{\epsilon}(\xi,\alpha,k) = -\frac{\tilde{q}(\xi)}{\xi^2 - 2k\alpha\cdot\xi} - \frac{1}{(2\pi)^3}\frac{1}{\xi^2 - 2k\alpha\cdot\xi}\tilde{q} * \tilde{\epsilon}. \tag{13}$$

An essential ingredient of our proof in section 3 is the following lemma, proved by the author (see its proof in [9], p 262, or in [8]):

Lemma 2.1. *If $A_j(\beta,\alpha,k)$ is the scattering amplitude corresponding to the potential q_j, then*

$$-4\pi[A_1(\beta,\alpha,k) - A_2(\beta,\alpha,k)] = \int_{B_1} [q_1(x) - q_2(x)]u_1(x,\alpha,k)u_2(x,-\beta,k)\,\mathrm{d}x, \tag{14}$$

where u_j is the scattering solution corresponding to q_j.

Consider an algebraic variety $\mathcal{M}$ in $\mathbb{C}^3$ defined by the equation

$$\theta \cdot \theta = 1, \qquad \theta \cdot \theta := \theta_1^2 + \theta_2^2 + \theta_3^2, \qquad \theta_j \in \mathbb{C}. \tag{15}$$

This is a non-compact variety, intersecting $\mathbb{R}^3$ over the unit sphere S^2.

Let $R_+ = [0, \infty)$. The following result is proved in [10], p 62.

Lemma 2.2. *If assumption (A) holds, then the scattering amplitude $A(\beta, \alpha, k)$ is a restriction to $S^2 \times S^2 \times R_+$ of a function $A(\theta', \theta, k)$ on $\mathcal{M} \times \mathcal{M} \times \mathbb{C}$, analytic on $\mathcal{M} \times \mathcal{M}$ and meromorphic on $\mathbb{C}$, $\theta', \theta \in \mathcal{M}$, $k \in \mathbb{C}$.*

The scattering solution $u(x, \alpha, k)$ is a meromorphic function of k in $\mathbb{C}$, analytic in $\mathrm{Im}k \geqslant 0$, except, possibly, at the points $k = \mathrm{i}k_j$, $1 \leqslant j \leqslant n$.

We need the notion of the Radon transform (see, e.g., [11]):

$$\hat{f}(\beta, \lambda) := \int_{\beta \cdot x = \lambda} f(x)\, \mathrm{d}\sigma, \tag{16}$$

where $\mathrm{d}\sigma$ is the element of the area of the plane $\beta \cdot x = \lambda$, $\beta \in S^2$, $\lambda = \text{const}$. One has (see [11], pp 12, 15)

$$\int_{B_a} f(x)\, \mathrm{d}x = \int_{-a}^{a} \hat{f}(\beta, \lambda)\, \mathrm{d}\lambda, \tag{17}$$

$$\int_{B_a} \mathrm{e}^{\mathrm{i}k\beta \cdot x} f(x)\, \mathrm{d}x = \int_{-a}^{a} \mathrm{e}^{\mathrm{i}k\lambda} \hat{f}(\beta, \lambda)\, \mathrm{d}\lambda, \tag{18}$$

$$\hat{f}(\beta, \lambda) = \hat{f}(-\beta, -\lambda). \tag{19}$$

Finally, we need a Phragmen–Lindelöf lemma, which is proved in [1], p 69, and in [2].

Lemma 2.3. *Let $f(z)$ be holomorphic inside an angle $\mathcal{A}$ of opening $< \pi$; $|f(z)| \leqslant c_1\, \mathrm{e}^{c_2|z|}$, $z \in \mathcal{A}$; $|f(z)| \leqslant M$ on the boundary of $\mathcal{A}$; and f is continuous up to the boundary of $\mathcal{A}$. Then $|f(z)| \leqslant M, \forall z \in \mathcal{A}$.*

3. Proof of theorem 1.1

We start with the observation that the scattering data in remark 1 determine uniquely the scattering data in theorem 1.1 by lemma 2.2.

Let us outline the ideas of the proof of theorem 1.1.

Assume that q_j, $j = 1, 2$, generate the same scattering data:

$$A_1(-\beta, \beta, k) = A_2(-\beta, \beta, k) \qquad \forall \beta \in S^2, \quad \forall k > 0,$$

and let

$$p(x) := q_1(x) - q_2(x).$$

Then by lemma 2.1, see equation (14), one gets

$$0 = \int_{B_a} p(x) u_1(x, \beta, k) u_2(x, \beta, k)\, \mathrm{d}x, \qquad \forall \beta \in S^2, \quad \forall k > 0. \tag{20}$$

By (9) and (11) one can rewrite (20) as

$$\int_{B_a} \mathrm{e}^{2\mathrm{i}k\beta \cdot x} [1 + \epsilon(x, k)] p(x)\, \mathrm{d}x = 0, \qquad \forall \beta \in S^2, \quad \forall k > 0, \tag{21}$$

where

$$\epsilon(x,k) := \epsilon := \epsilon_1(x,k) + \epsilon_2(x,k) + \epsilon_1(x,k)\epsilon_2(x,k).$$

By lemma 2.2, relations (20) and (21) hold for complex k,

$$k = \frac{\kappa + \mathrm{i}\eta}{2}, \qquad \kappa + \mathrm{i}\eta \neq 2\mathrm{i}k_j, \quad \eta \geqslant 0, \tag{22}$$

in particular, for $\eta > k_n$, $\kappa \in \mathbb{R}$. Using formulas (3)–(4), one derives from (21) the relation

$$\tilde{p}((\kappa + \mathrm{i}\eta)\beta) + \frac{1}{(2\pi)^3}(\tilde{\epsilon} * \tilde{p})((\kappa + \mathrm{i}\eta)\beta) = 0 \qquad \forall\beta \in S^2, \quad \forall\kappa \in \mathbb{R}, \quad \eta > k_n, \tag{23}$$

where the notation $(f * g)(z)$ means that the convolution $f * g$ is calculated at the argument $(\kappa + \mathrm{i}\eta)\beta$.

One has

$$\sup_{\beta\in S^2} |\tilde{\epsilon} * \tilde{p}| := \sup_{\beta\in S^2} \left| \int_{\mathbb{R}^3} \tilde{\epsilon}((\kappa + \mathrm{i}\eta)\beta - s)\tilde{p}(s)\,\mathrm{d}s \right| \leqslant \nu(\kappa,\eta) \sup_{s\in\mathbb{R}^3} |\tilde{p}(s)|, \tag{24}$$

where

$$\nu(\kappa,\eta) := \sup_{\beta\in S^2} \int_{\mathbb{R}^3} |\tilde{\epsilon}((\kappa + \mathrm{i}\eta)\beta - s)|\,\mathrm{d}s.$$

We will prove that if $\eta = \eta(\kappa) = O(\ln\kappa)$, then the following inequality holds:

$$0 < \nu(\kappa,\eta(\kappa)) < 1, \qquad \kappa \to \infty. \tag{25}$$

If one proves that

$$\sup_{\beta\in S^2} |\tilde{p}((\kappa + \mathrm{i}\eta(\kappa))\beta)| \geqslant \sup_{s\in\mathbb{R}^3} |\tilde{p}(s)|, \qquad \kappa \to \infty, \tag{26}$$

then it follows from (23)–(26) that $\tilde{p}(s) = 0$, so $p(x) = 0$, and theorem 1.1 is proved. Indeed, it follows from (23) and (26) that

$$\sup_{s\in\mathbb{R}^3} |\tilde{p}(s)| \leqslant \frac{1}{(2\pi)^3}\nu(k,\eta) \sup_{s\in\mathbb{R}^3} |\tilde{p}(s)|.$$

If (25) holds, then the above equation implies that $\tilde{p} = 0$. This and the injectivity of the Fourier transform imply that $p = 0$.

This completes the outline of the proof of theorem 1.1.

Let us now establish estimates (25) and (26).

We assume that $p(x) \not\equiv 0$, because otherwise there is nothing to prove. If $p(x) \not\equiv 0$, then

$$\max_{s\in\mathbb{R}^3} |\tilde{p}(s)| := \mathcal{P} \neq 0.$$

Lemma 3.1. *If assumption (A) holds and $\mathcal{P} \neq 0$, then*

$$\limsup_{\eta\to\infty} \max_{\beta\in S^2} |\tilde{p}((\kappa + \mathrm{i}\eta)\beta)| = \infty, \tag{27}$$

where $\kappa > 0$ is arbitrary but fixed. For any $\kappa > 0$ there is a $\eta = \eta(\kappa)$, such that

$$\max_{\beta\in S^2} |\tilde{p}((\kappa + \mathrm{i}\eta(\kappa))\beta)| = \mathcal{P}, \tag{28}$$

where the number $\mathcal{P} := \max_{s\in\mathbb{R}^3} |\tilde{p}(s)|$, and

$$\eta(\kappa) = O(\ln\kappa) \qquad \text{as} \quad \kappa \to +\infty. \tag{29}$$

Proof of lemma 3.1. By formula (18) one gets

$$\tilde{p}((\kappa+\mathrm{i}\eta)\beta)=\int_{B_a}p(x)\,\mathrm{e}^{\mathrm{i}(\kappa+\mathrm{i}\eta)\beta\cdot x}\,\mathrm{d}x=\int_{-a}^{a}\mathrm{e}^{\mathrm{i}\kappa\lambda-\eta\lambda}\hat{p}(\beta,\lambda)\,\mathrm{d}\lambda. \tag{30}$$

The function $\hat{p}(\beta,\lambda)$ satisfies (19). Therefore,

$$\max_{\beta\in S^2}|\tilde{p}((\kappa+\mathrm{i}\eta(\kappa))\beta)|=\max_{\beta\in S^2}|\tilde{p}((\kappa-\mathrm{i}\eta(\kappa))\beta)|. \tag{31}$$

Indeed,

$$\begin{aligned}\max_{\beta\in S^2}|\tilde{p}((\kappa+\mathrm{i}\eta(\kappa))\beta)&=\max_{\beta\in S^2}\left|\int_{-a}^{a}\mathrm{e}^{\mathrm{i}\kappa\lambda-\eta\lambda}\hat{p}(\beta,\lambda)\,\mathrm{d}\lambda\right|\\&=\max_{\beta\in S^2}\left|\int_{-a}^{a}\mathrm{e}^{-\mathrm{i}\kappa\mu+\eta\mu}\hat{p}(\beta,-\mu)\,\mathrm{d}\mu\right|\\&=\max_{\beta'\in S^2}\left|\int_{-a}^{a}\mathrm{e}^{-\mathrm{i}\kappa\mu+\eta\mu}\hat{p}(-\beta',-\mu)\,\mathrm{d}\mu\right|\\&=\max_{\beta'\in S^2}\left|\int_{-a}^{a}\mathrm{e}^{-\mathrm{i}\kappa\mu+\eta\mu}\hat{p}(\beta',\mu)\,\mathrm{d}\mu\right|\\&=\max_{\beta\in S^2}|\tilde{p}((\kappa-\mathrm{i}\eta)\beta)|.\end{aligned} \tag{32}$$

Here we took into account that $\hat{p}(\beta,\lambda)$ is a real-valued function because $q_j(x)$ are real valued. Therefore,

$$\left|\int_{-a}^{a}\mathrm{e}^{-\mathrm{i}\kappa\mu+\eta\mu}\hat{p}(\tilde{\beta},\mu)\,\mathrm{d}\mu\right|=\left|\int_{-a}^{a}\mathrm{e}^{\mathrm{i}\kappa\mu+\eta\mu}\hat{p}(\tilde{\beta},\mu)\,\mathrm{d}\mu\right|=\max_{\beta\in S^2}|\tilde{p}((\kappa-\mathrm{i}\eta)\beta)|.$$

If $p(x)\not\equiv 0$, then (30) and (31) imply (27), as follows from lemma 2.3. Let us give a detailed argument.

Consider the function h of the complex variable $z:=\kappa+\mathrm{i}\eta$:

$$h:=h(z,\beta):=\int_{-a}^{a}\mathrm{e}^{\mathrm{i}z\lambda}\hat{p}(\beta,\lambda)\,\mathrm{d}\lambda. \tag{33}$$

If (27) is false, then

$$|h(z,\beta)|\leqslant c\quad\forall z=\kappa+\mathrm{i}\eta,\qquad \eta\geqslant 0,\quad \forall\beta\in S^2, \tag{34}$$

where $\kappa\geqslant 0$ is an arbitrary fixed number, and the constant $c>0$ does not depend on β and η.

Thus, $|h|$ is bounded on the ray $\{\kappa=0,\eta\geqslant 0\}$, which is part of the boundary of the right angle $\mathcal{A}$, and the other part of its boundary is the ray $\{\kappa\geqslant 0,\eta=0\}$. Let us check that $|h|$ is bounded on this ray also.

One has

$$|h(\kappa,\beta)|\leqslant\left|\int_{-a}^{a}\mathrm{e}^{\mathrm{i}\kappa\lambda}\hat{p}(\beta,\lambda)\,\mathrm{d}\lambda\right|\leqslant\int_{-a}^{a}|\hat{p}(\beta,\lambda)|\,\mathrm{d}\lambda\leqslant c, \tag{35}$$

where c stands for *various* constants. From (34)–(35) it follows that on the boundary of the right angle $\mathcal{A}$, namely, on the two rays $\{\kappa\geqslant 0,\eta=0\}$ and $\{\kappa=0,\eta\geqslant 0,\}$, the entire function $h(z,\beta)$ is bounded, $|h(z,\beta)|\leqslant c$, and inside $\mathcal{A}$ this function satisfies the estimate

$$|h(z,\beta)|\leqslant\mathrm{e}^{|\eta|a}\int_{-a}^{a}|\hat{p}(\beta,\lambda)|\,\mathrm{d}\lambda\leqslant c\,\mathrm{e}^{|\eta|a}. \tag{36}$$

Therefore, by lemma 2.3, $|h(z,\beta)|\leqslant c$ in the whole angle $\mathcal{A}$.

By (31) the same argument is applicable to the remaining three right angles, the union of which is the whole complex z-plane $\mathbb{C}$. Therefore,

$$\sup_{z\in\mathbb{C},\beta\in S^2} |h(z,\beta)| \leqslant c. \tag{37}$$

This implies that $h(z,\beta) = c$.

Since $\hat{p}(\beta,\lambda) \in L^1(-a,a)$, the relation

$$\int_{-a}^{a} \mathrm{e}^{\mathrm{i}z\lambda}\hat{p}(\beta,\lambda)\,\mathrm{d}\lambda = c \quad \forall z \in \mathbb{C}, \tag{38}$$

implies that $c = 0$, so $\hat{p}(\beta,\lambda) = 0$. Therefore $p(x) = 0$, contrary to our assumption. Consequently, relation (27) is proved.

Relation (28) follows from (27) because for large η the left-hand side of (28) is larger than $\mathcal{P}$ due to (27), while for $\eta = 0$ the left-hand side of (28) is not larger than $\mathcal{P}$ by the definition of the Fourier transform. □

Let us derive estimate (29).

From the assumption $p(x) \in H_0^\ell(B_a)$ it follows that

$$|\tilde{p}((\kappa + \mathrm{i}\eta)\beta)| \leqslant c\frac{\mathrm{e}^{a|\eta|}}{(1+\kappa^2+\eta^2)^{\ell/2}}. \tag{39}$$

This inequality is established in Lemma 3.2, below.

The right-hand side of this inequality is of the order $O(1)$ as $\kappa \to \infty$ if and only if $|\eta| = O(\ln\kappa)$, which is relation (29).

Lemma 3.2. *If $p \in H_0^\ell(B_a)$ then estimate (39) holds.*

Proof. Consider $\partial_j p := \frac{\partial p}{\partial x_j}$. One has

$$\begin{aligned}\left|\int_{B_a} \partial_j p\, \mathrm{e}^{\mathrm{i}(\kappa+\mathrm{i}\eta)\beta\cdot x}\,\mathrm{d}x\right| &= \left|-\mathrm{i}(\kappa+\mathrm{i}\eta)\beta_j \int_{B_a} p(x)\mathrm{e}^{\mathrm{i}(\kappa+\mathrm{i}\eta)\beta\cdot x}\,\mathrm{d}x\right| \\ &= (\kappa^2+\eta^2)^{1/2}|\tilde{p}((\kappa+\mathrm{i}\eta)\beta)|. \end{aligned} \tag{40}$$

The left-hand side of the above formula admits the following estimate:

$$\left|\int_{B_a} \partial_j p\, \mathrm{e}^{\mathrm{i}(\kappa+\mathrm{i}\eta)\beta\cdot x}\,\mathrm{d}x\right| \leqslant c\,\mathrm{e}^{|\eta|a},$$

where the constant $c > 0$ is proportional to $\|\partial_j p\|_{L^2(B_a)}$. Therefore,

$$|\tilde{p}((\kappa+\mathrm{i}\eta)\beta)| \leqslant c[1+(\kappa^2+\eta^2)]^{-1/2}\,\mathrm{e}^{|\eta|a}. \tag{41}$$

Repeating this argument one gets estimate (39).

Estimate (41) implies that if estimate (29) holds and $\kappa \to \infty$, then the quantity $\sup_{\beta\in S^2} |\tilde{p}((\kappa+\mathrm{i}\eta)\beta)|$ remains bounded as $\kappa \to \infty$.

If η is fixed and $\kappa \to \infty$, then $\sup_{\beta\in S^2} |\tilde{p}((\kappa+\mathrm{i}\eta)\beta)| \to 0$ by the Riemann–Lebesgue lemma. This and (27) imply the existence of $\eta = \eta(\kappa)$, such that (28) holds, and, consequently, (26) holds. This $\eta(\kappa)$ satisfies (29) because $\mathcal{P}$ is bounded. □

To complete the proof of theorem 1.1 one has to establish estimate (25). This estimate will be established if one proves the following:

$$\lim_{\kappa\to\infty} \nu(\kappa) := \lim_{\kappa\to\infty} \nu(\kappa, \eta(\kappa)) = 0, \tag{42}$$

where $\eta(\kappa) = O(\ln \kappa)$ and

$$\nu(\kappa, \eta) = \sup_{\beta\in S^2} \int_{\mathbb{R}^3} |\tilde{\epsilon}((\kappa + \mathrm{i}\eta)\beta - s)|\, \mathrm{d}s. \tag{43}$$

Our argument is valid for ϵ_1, ϵ_2 and $\epsilon_1\epsilon_2$, so we will use the letter ϵ and equation (13) for $\tilde{\epsilon}$.

We prove that (13) can be solved by iterations if $|k + \mathrm{i}\eta|$ is sufficiently large and $\eta = \eta(k)$, because for such k the operator in this equation has small norm. Therefore, the estimate of the solution $\tilde{\epsilon}$ to this equation is similar to the estimate of the free term of this equation. Thus, it is sufficient to check estimates (42)–(43) for the function $\tilde{q}(\xi)(\xi^2 - 2k\beta\cdot\xi)^{-1}$, with $2k$ replaced by $\kappa + \mathrm{i}\eta$, because equation (12) has an operator

$$T\epsilon = \int_{B_a} G(x - y, k)q(y)\epsilon(y, k)\, \mathrm{d}y,$$

and the norm $\|T^2\|$ (in the space $C(B_a)$ of functions with the sup norm) tends to zero as $\kappa = 2\,\mathrm{Re}\, k \to \infty$. Consequently, equation (12) can be solved by iterations and the main term in its solution, as $|\kappa + \mathrm{i}\eta| \to \infty$, $\eta \geqslant 0$, is the free term in this equation. The same is true for the Fourier transform of equation (12), i.e. for equation (13).

Let us estimate the integral

$$\begin{aligned} I &= \sup_{\beta\in S^2} \int_{\mathbb{R}^3} \frac{|\tilde{q}((\kappa + \mathrm{i}\eta)\beta - s)|\mathrm{d}s}{|[(\kappa + \mathrm{i}\eta)\beta - s)^2 - (\kappa + \mathrm{i}\eta)\beta\cdot((\kappa + \mathrm{i}\eta)\beta - s)]|} \\ &\leqslant c \sup_{\beta\in S^2} \mathrm{e}^{|\eta|a} \int_{\mathbb{R}^3} \frac{\mathrm{d}s}{|s^2 - (\kappa + \mathrm{i}\eta)\beta\cdot s|[1 + (\kappa\beta - s)^2 + \eta^2]^{\ell/2}} \\ &:= c\, \mathrm{e}^{|\eta|a} J. \end{aligned} \tag{44}$$

Here estimate (39) was used.

Let us write the integral J in the spherical coordinates with the x_3-axis directed along the vector β, $|s| = r$, $\beta\cdot s = r\cos\theta := rt$, $-1 \leqslant t \leqslant 1$. Let

$$\gamma := \kappa^2 + \eta^2. \tag{45}$$

Then

$$\begin{aligned} J &\leqslant 2\pi \int_0^\infty \mathrm{d}r\, r \int_{-1}^1 \frac{\mathrm{d}t}{[(r - \kappa t)^2 + \eta^2 t^2]^{1/2}(1 + \gamma + r^2 - 2r\kappa t)^{\ell/2}} \\ &:= 2\pi \int_0^\infty \mathrm{d}r\, r B(r), \end{aligned} \tag{46}$$

where

$$B := B(r) = B(r, \kappa, \eta) := \int_{-1}^1 \frac{\mathrm{d}t}{[(r - \kappa t)^2 + \eta^2 t^2]^{1/2}(1 + \gamma + r^2 - 2r\kappa t)^{\ell/2}}. \tag{47}$$

If $t \in [-1, 1]$, then

$$1 + \gamma + r^2 - 2r\kappa t \geqslant 1 + \gamma^2 + r^2 - 2r\kappa = 1 + \eta^2 + (r - \kappa)^2. \tag{48}$$

Thus,

$$B \leqslant \frac{1}{[1+\eta^2+(r-\kappa)^2]^{\ell/2}} \frac{1}{\sqrt{\gamma}} \int_{-1}^{1} \frac{\mathrm{d}t}{\left[\left(t-\frac{r\kappa}{\gamma}\right)^2+\frac{\eta^2 r^2}{\gamma^2}\right]^{1/2}}$$

$$= \frac{1}{\sqrt{\gamma}[1+\eta^2+(r-\kappa)^2]^{\ell/2}} \left| \ln \left| \frac{1-\frac{r\kappa}{\gamma}+\sqrt{\left(1-\frac{r\kappa}{\gamma}\right)^2+\frac{\eta^2 r^2}{\gamma^2}}}{\sqrt{\left(1+\frac{r\kappa}{\gamma}\right)^2+\frac{\eta^2 r^2}{\gamma^2}}-1-\frac{r\kappa}{\gamma}} \right| \right|. \tag{49}$$

Consequently,

$$J \leqslant \frac{2\pi}{\sqrt{\gamma}} \int_0^\infty \frac{\mathrm{d}r\, r}{[1+\eta^2+(r-\kappa)^2]^{\ell/2}} \left| \ln \left| \frac{1-\frac{r\kappa}{\gamma}+\sqrt{\left(1-\frac{r\kappa}{\gamma}\right)^2+\frac{\eta^2 r^2}{\gamma^2}}}{\sqrt{\left(1+\frac{r\kappa}{\gamma}\right)^2+\frac{\eta^2 r^2}{\gamma^2}}-1-\frac{r\kappa}{\gamma}} \right| \right|. \tag{50}$$

The integral in (50) converges: as $r \to \infty$ the ratio under the logarithm sign tends to 1, and the factor in front of the logarithm is $O(r^{-(\ell-1)})$ as $r \to \infty$. Since $\ell > 2$, the integral in (50) converges.

The modulus of the logarithmic term in (50) behaves asymptotically, as $r \to 0$, like $\left|\ln\left(\frac{r^2\kappa^2}{\gamma^2}\right)\right|$. Thus, $\lim_{r\to 0} r\left|\ln\left(\frac{r^2\kappa^2}{\gamma^2}\right)\right| = 0$ for every fixed $\kappa > 0$, and this limit is uniform with respect to κ as $\kappa \to \infty$ if $\eta = O(\ln\kappa)$. Therefore, the integrand in (50) is defined for $r = 0$ to be zero by continuity.

As $\gamma = \eta^2+\kappa^2 \to \infty$ and $\eta = O(\ln\kappa)$, the integrand in (50) tends to zero for every fixed $r \geqslant 0$, and (50) implies

$$J \leqslant o\left(\frac{1}{\sqrt{\gamma}}\right), \qquad \gamma \to \infty. \tag{51}$$

Consequently, (44) implies

$$I \leqslant cr^{|\eta|a} o\left(\frac{1}{\sqrt{\kappa^2+\eta^2}}\right), \qquad \kappa \to \infty, \quad \eta = O(\ln\kappa). \tag{52}$$

Therefore,

$$\lim_{\kappa\to\infty,\eta=O(\ln\kappa)} I = 0. \tag{53}$$

This implies estimate (42). Theorem 1.1 is proved. □

Remark 2. Similarly one can prove that the data $A(\beta, \alpha_0, k)$, $\forall\beta \in S^2$, $\forall k > 0$, and a fixed $\alpha = \alpha_0 \in S^2$ determine q uniquely if assumption (A) holds.

Appendix. Estimate of the norm of the operator T^2

Let

$$Tf := \int_{B_a} G(x-y, \kappa+\mathrm{i}\eta) q(y) f(y)\, \mathrm{d}y. \tag{A.1}$$

Assume $q \in H_0^\ell(B_a)$, $\ell > 2$, $f \in C(B_a)$. Our goal is to prove that equation (12) can be solved by iterations for all sufficiently large κ.

Consider T as an operator in $C(B_a)$. One has

$$T^2 f = \int_{B_a} \mathrm{d}z\, G(x-z, \kappa+\mathrm{i}\eta) q(z) \int_{B_a} G(z-y, \kappa+\mathrm{i}\eta) q(y) f(y)\, \mathrm{d}y$$

$$= \int_{B_a} \mathrm{d}y\, f(y) q(y) \int_{B_a} \mathrm{d}z\, q(z) G(x-z, \kappa+\mathrm{i}\eta) G(z-y, \kappa+\mathrm{i}\eta). \tag{A.2}$$

Let us estimate the integral

$$I(x,y) := \int_{B_a} G(x-z,\kappa+i\eta)G(z-y,\kappa+i\eta)q(z)\,dz$$
$$= \int_{B_a} \frac{e^{i(\kappa+i\eta)[|x-z|-\beta\cdot(x-z)+|z-y|-\beta\cdot(z-y)]}}{16\pi^2|x-z||z-y|}q(z)\,dz$$
$$= \frac{1}{16\pi^2}\int_{B_a} \frac{e^{i(\kappa+i\eta)[|x-z|+|z-y|-\beta\cdot(x-y)]}}{|x-z||z-y|}q(z)\,dz$$
$$:= \frac{e^{-i(\kappa+i\eta)\beta\cdot(x-y)}}{16\pi^2}I_1(x,y). \tag{A.3}$$

Let us use the following coordinates (see [10], p 391):

$$z_1 = \ell st + \frac{x_1+y_1}{2}, \qquad z_2 = \ell\sqrt{(s^2-1)(1-t^2)}\cos\psi + \frac{x_2+y_2}{2}, \tag{A.4}$$

$$z_3 = \ell\sqrt{(s^2-1)(1-t^2)}\sin\psi + \frac{x_3+y_3}{2}. \tag{A.5}$$

The Jacobian J of the transformation $(z_1,z_2,z_3)\to(\ell,t,\psi)$ is

$$J = \ell^3(s^2-t^2), \tag{A.6}$$

where

$$\ell = \frac{|x-y|}{2}, \qquad |x-z|+|z-y| = 2\ell s, \qquad |x-z|-|z-y| = 2\ell t, \tag{A.7}$$

$$|x-z||z-y| = 4\ell^2(s^2-t^2), \qquad 0\leqslant\psi<2\pi, \qquad t\in[-1,1], \quad s\in[1,\infty). \tag{A.8}$$

One has

$$I_1 = \ell\int_a^\infty e^{2i(\kappa+i\eta)\ell s}Q(s)\,ds, \tag{A.9}$$

where

$$Q(s) := Q\left(s,\ell,\frac{x+y}{2}\right) = \int_0^{2\pi} d\psi\int_{-1}^1 dt\, q\left(z\left(s,t,\psi;\ell,\frac{x+y}{2}\right)\right), \tag{A.10}$$

and the function $Q(s)\in H_0^2(\mathbb{R}^3)$ for any fixed x,y. Therefore, an integration by parts in (A.9) yields the following estimate:

$$|I_1| = O\left(\frac{1}{|\kappa+i\eta|}\right), \qquad |\kappa+i\eta|\to\infty. \tag{A.11}$$

From (A.2), (A.3) and (A.11) one gets

$$\|T^2\| = O\left(\frac{1}{\sqrt{\gamma}}\right), \qquad \gamma := \kappa^2+\eta^2\to\infty. \tag{A.12}$$

Therefore, integral equation (12), with k replaced by $\frac{\kappa+i\eta}{2}$, can be solved by iterations if γ is sufficiently large and $\eta\geqslant 0$. Consequently, integral equation (13) can be solved by iterations. Thus, estimate (42) holds if such an estimate holds for the free term in equation (13), that is, for the function $\frac{\tilde{q}}{\xi^2-(\kappa+i\eta)\beta\cdot\xi}$, namely, if estimate (53) holds.

References

[1] Levin B 1980 *Distribution of Zeros of Entire Functions* (Providence, RI: American Mathematical Society)
[2] Polya G and Szegö G 1983 *Problems and Theorems in Analysis* vol 1 (Berlin: Springer) problem III.6.5.322
[3] Ramm A G 2009 Inverse scattering with non-overdetermined data *Phys. Lett.* A **373** 2988–91
[4] Ramm A G 1964 On the analytic continuation of the solution of the Schrödinger equation in the spectral parameter and the behavior of the solution to the nonstationary problem as $t \to \infty$ *Uspechi Mat. Nauk* **19** 192–4
[5] Ramm A G 1968 Some theorems on analytic continuation of the Schrödinger operator resolvent kernel in the spectral parameter *Izv. Acad. Nauk Arm. SSR, Math.* **3** 443–64
[6] Ramm A G 2005 *Random Fields Estimation* (Singapore: World Scientific)
[7] Ramm A G 1988 Recovery of the potential from fixed energy scattering data *Inverse Problems* **4** 877–86
Ramm A G 1989 Recovery of the potential from fixed energy scattering data *Inverse Problems* **5** 255
[8] Ramm A G 2002 Stability of solutions to inverse scattering problems with fixed-energy data *Milan J. Math.* **70** 97–161
[9] Ramm A G 2005 *Inverse Problems* (New York: Springer)
[10] Ramm A G 1986 *Scattering by Obstacles* (Dordrecht: Reidel)
[11] Ramm A G and Katsevich A I 1996 *The Radon Transform and Local Tomography* (Boca Raton, FL: CRC Press)
[12] Saito Y 1986 An approximation formula in inverse scattering problem *J. Math. Phys.* **27** 1145–53